The relation between GT_i-spaces and fuzzy proximity spaces, G-compact spaces, fuzzy uniform spaces

Fatma Bayoumi and Ismail Ibedou

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

E-mail: fatma_bayoumi@hotmail.com

Abstract

In this paper, we study the relation between the fuzzy separation axioms, which had been introduced by the authors in 2001, and the fuzzy proximity defined by Katsaras in 1980. We study also the relation between our fuzzy separation axioms and the G-compactness defined by Gähler in 1995. Moreover, we show here the relation between these fuzzy separation axioms and the fuzzy uniform structures introduced and studied by Gähler and the first author in 1998.

Keywords: Fuzzy filters; Fuzzy neighborhood filters; GT_i-separation axioms; GT_i-spaces; Fuzzy proximity spaces; G-compact spaces; Fuzzy uniform spaces; Separated fuzzy uniform spaces.

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0. Introduction

In [1, 2] we had introduced a new kind of fuzzy separation axioms for fuzzy topological spaces related to the usual points and ordinary subsets. These axioms are defined, analogously to the separation axioms in the classical case ([6]), using the notion of fuzzy neighborhood filter which had been introduced by Gähler in [9]. Denote by GT_i for these axioms and by GT_i-spaces for the fuzzy topological spaces which fulfill the axioms GT_i. We had studied the cases $i = 0, 1, 2, 3, 4$.

Our fuzzy separation axioms GT_i and our GT_i-spaces, $i = 0, 1, 2, 3, 4$ fulfill many properties analogous to the usual ones. Some of these properties had studied in [1, 2, 3]. This paper is devoted to study further properties for these axioms. We study here the relation between the GT_i-spaces and the fuzzy proximity spaces, G-compact spaces and the fuzzy uniform spaces.

In Section 1 of this paper some definitions and notations, related to fuzzy sets, fuzzy topologies, fuzzy filters and fuzzy separation axioms GT_i are given.

Section 2 is devoted to study the relation between the fuzzy proximity defined by Katsaras in [12] and our fuzzy separation axioms. It will be shown that any fuzzy proximity is separated if and only if the associated fuzzy topology is GT_0 and to each fuzzy proximity is associated a regular fuzzy topology in our sense. Moreover, for each normal fuzzy topological space (X, τ) the binary relation on L^X defined by means of the closure operator cl_τ of τ, in equation (2.7), is a fuzzy proximity on X and conversely, to each fuzzy proximity δ, which has a closure operator cl_δ fulfills the binary relation given in (2.7), is associated a normal fuzzy topology.

There is a good notion of fuzzy compactness, called G-Compactness, had been introduced and studied by Gähler in [9]. This notion fulfills main properties. For example: It fulfills the Tychonoff Theorem. The notion of G-Compactess is defined using the fuzzy filters, and so it is possible to make a relation between this compactness and our fuzzy separation axioms. This relation will be shown in Section 3. It will be shown that each G-Compact subset of GT_2-space is closed and that each G-Compact GT_2-space is GT_4-space and moreover that if (X, τ_2) is a G-Compact space finer than a GT_2-space (X, τ_1), then (X, τ_1) is homeomorphic to (X, τ_2).

A notion of fuzzy uniform structure had been introduced and studied by Gähler et all in [10]. A fuzzy uniform structure $\mathcal{U}$ on a set X in sense of [10] is defined, analogously to Weil's definition of a uniform structure ([14]), as a special fuzzy filter on the product set $X \times X$. We introduce in the last section of this paper the notion of separated fuzzy uniform space and we then show the relation between these separated fuzzy uniform spaces and the GT_i-spaces. It will be shown that the fuzzy uniform spaces are separated if and only if the induced fuzzy topological spaces are GT_0. As an example for the application of fuzzy sets and fuzzy topology one could mention the recent work on the connection between quantum gravity and the Cantor space $C = 2^N$ of descripte set theory [16]. We note that replacing $A = 2 = [0, 1]$ by $A = [0 - 2] = (1 + \sqrt{5})/2$ one obtain what may be called fuzzy Cantor space used in E-Infinity by Elnaschie.

1. Notations and Preliminaries

In the following consider L is a complete chain with different least and last elements 0 and 1, respectively. Let $L_0 = L \setminus \{0\}$. For each set X let L^X denote the set of all L-fuzzy subsets (or simply, fuzzy subsets) of X, that is, of all mappings $f : X \to L$. Assume that an order-reversing involution $\alpha \mapsto \alpha'$ of L is fixed. For each fuzzy set $f \in L^X$, let f' denote the complement of f defined by: $f'(x) = f(x)'$ for all $x \in X$. For each $\alpha \in L$ let $\bar{\alpha}$ denote the constant fuzzy subset of X with value α. For all $x \in X$ and $\alpha \in L_0$, the fuzzy subset x_α of X whose value α at x and 0 otherwise is called a *fuzzy point* in X. By a *fuzzy topology* on X is meant ([5, 11]) a subset τ of L^X which contains the constant fuzzy sets $\overline{0}$ and $\overline{1}$ and is closed with respect to the finite infima and arbitrary suprema. The pair (X, τ) is called a *fuzzy topological space*. The elements of τ are called open fuzzy subsets of X and the complements of the open fuzzy sets are called closed fuzzy subsets of X. Denote by τ' for the class of all closed fuzzy subsets of X. A fuzzy topology τ is called *stratified* ([13]) if $\overline{\alpha} \in \tau$ for each $\alpha \in L$.

Let τ_1 and τ_2 be fuzzy topologies on X. Then τ_1 ia said to be *finer* than τ_2 or τ_2 is said to be *coarser* than τ_1 if $\tau_1 \supseteq \tau_2$. The interior $\mathrm{int}_\tau f$ of a fuzzy subset f of X, with respect to the fuzzy topology τ on X, is the greatest open fuzzy subset of X less than or equal to f. Moreover, the closure $\mathrm{cl}_\tau f$ of f, with respect to τ, is the smallest closed fuzzy subset of X greater than or equal to f.

For more informations on the fuzzy sets, and fuzzy topological spaces we refer to [5, 11, 15].

Fuzzy filters. A mapping $\mathcal{M} : L^X \to L$ is said to be a *fuzzy filter* on a non-empty set X ([8]) if the following conditions are fulfilled:

(F1) $\mathcal{M}(\bar{\alpha}) \leq \alpha$ for all $\alpha \in L$ and $\mathcal{M}(\,\bar{1}\,) = 1$.

(F2) $\mathcal{M}(f \wedge g) = \mathcal{M}(f) \wedge \mathcal{M}(g)$ for all $f, g \in L^X$.

A fuzzy filter $\mathcal{M}$ is called *homogeneous* ([7]) if $\mathcal{M}(\bar{\alpha}) = \alpha$ for all $\alpha \in L$. Denote by $\mathcal{F}_L X$ and $\mathsf{F}_L X$ for the sets of all fuzzy filters and of all homogeneous fuzzy filters on X, respectively. For each $x \in X$, the mapping $\dot{x} : L^X \to L$ defined by $\dot{x}(f) = f(x)$ for all $f \in L^X$ is a homogeneous fuzzy filter on X.

If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on a set X, $\mathcal{M}$ is said to be *finer* than $\mathcal{N}$ or $\mathcal{N}$ is said to be *coarser* than $\mathcal{M}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(f) \geq \mathcal{N}(f)$ holds for all $f \in L^X$. By $\mathcal{M} \not\leq \mathcal{N}$ we denote that $\mathcal{M}$ is not finer than $\mathcal{N}$. If L is a complete chain, then

$$\mathcal{M} \not\leq \mathcal{N} \iff \text{ there is } f \in L^X \text{ such that } \mathcal{M}(f) < \mathcal{N}(f).$$

Let A be a set of fuzzy filters $\mathcal{M}$ on a set X. The supremum $\bigvee\limits_{\mathcal{M} \in A} \mathcal{M}$ of A with respect to the finer relation of fuzzy filters exists if and only if A is non-empty. Whereas the infimum $\bigwedge\limits_{\mathcal{M} \in A} \mathcal{M}$ of A does not exist, in general, as a fuzzy filter. The infimum of A exists if A is bounded below. See the definitions of $\bigvee\limits_{\mathcal{M} \in A} \mathcal{M}$ and $\bigwedge\limits_{\mathcal{M} \in A} \mathcal{M}$ in [8].

Proposition 1.1 [8] *Let A be a set of fuzzy filters on X. Then the infimum of A with respect to the finer relation of fuzzy filters exists if and only if for each non-empty finite subset $\{\mathcal{M}_1, \ldots, \mathcal{M}_n\}$ of A we have*

$$\mathcal{M}_1(f_1) \wedge \cdots \wedge \mathcal{M}_n(f_n) \ \leq \ \sup(f_1 \wedge \cdots \wedge f_n)$$

for all $f_1, \ldots, f_n \in L^X$.

Fuzzy neighborhood filters at points and at sets. For each fuzzy topological space (X, τ) and each $x \in X$ the mapping $\mathcal{N}(x) : L^X \to L$ defined by

$$\mathcal{N}(x)(f) \ = \ \operatorname{int}_\tau f(x)$$

for all $f \in L^X$ is a fuzzy filter on X, called the *fuzzy neighborhood filter* of the space (X, τ) at x ([9]). The fuzzy neighborhood filters fulfill the following conditions:

(N1) $\dot{x} \leq \mathcal{N}(x)$ holds for all $x \in X$.

(N2) $\mathcal{N}(x)(y \mapsto \mathcal{N}(y)(f)) = \mathcal{N}(x)(f)$ for all $x \in X$ and $f \in L^X$.

The fuzzy neighborhood filter $\mathcal{N}(F)$ at a set $F \subseteq X$ is defined by means of $\mathcal{N}(x)$, $x \in F$ ([2]) as:

$$\mathcal{N}(F) \; = \; \bigvee_{x \in F} \mathcal{N}(x).$$

GT_i**-separation axioms and** GT_i**-spaces.** We had introduced some fuzzy separation axioms, called GT_i, in [1, 2] using the fuzzy neighborhood filters at points and at sets as follows.

A fuzzy topological space (X, τ) is called:

(1) GT_0 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ or $\dot{y} \nleq \mathcal{N}(x)$.

(2) GT_1 if for all $x, y \in X$ with $x \neq y$ we have $\dot{x} \nleq \mathcal{N}(y)$ and $\dot{y} \nleq \mathcal{N}(x)$.

(3) GT_2 or *Hausdorff* if for all $x, y \in X$ with $x \neq y$ we have $\mathcal{N}(x) \wedge \mathcal{N}(y)$ does not exist.

(4) *regular* if $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist for all $x \in X$, $F \subseteq X$ with $F = \mathrm{cl}_\tau F$ and $x \notin F$.

(5) GT_3 if it is regular and GT_1.

(6) *normal* if for all $F_1, F_2 \subseteq X$ with $F_1 = \mathrm{cl}_\tau F_1$, $F_2 = \mathrm{cl}_\tau F_2$ and $F_1 \cap F_2 = \emptyset$ we have $\mathcal{N}(F_1) \wedge \mathcal{N}(F_2)$ does not exist.

(7) GT_4 if it is normal and GT_1.

By GT_i-*space* we mean the fuzzy topological space which is GT_i.

2. Fuzzy Proximity Spaces

In this section we are going to study the relation between the fuzzy proximity δ defined in [12] and our fuzzy separation axioms GT_i. To study this relation we make at first a relation between the farness on fuzzy sets and the finer relation on fuzzy filters. We need in this case to define the fuzzy neighborhood filter $\mathcal{N}(f)$ and the homogeneous fuzzy filter $\dot{f}$ at a fuzzy subset f of a set X and we show some results for these fuzzy filters which will be used in this section.

In the following proposition we introduce the homogeneous fuzzy filter $\dot{f}$ for every $f \in L^X$.

Proposition 2.1 *Let X be a set and f be a fuzzy subset of X. Then the supremum of the homogeneous fuzzy filters $\dot{x}$*

$$\dot{f} = \bigvee_{0<f(x)} \dot{x} \tag{2.1}$$

is a homogeneous fuzzy filter *on X.*

Proof. Since $\dot{f} = \bigvee_{0<f(x)} \dot{x}$ for all $f \in L^X$, then $\dot{f}(g) = \bigwedge_{0<f(x)} g(x)$ for all $g \in L^X$ and hence $\dot{f}\,(\overline{\alpha}) = \bigwedge_{0<f(x)} \overline{\alpha}(x) = \alpha$ and

$$\dot{f}\,(g \wedge h) \;=\; \bigwedge_{0<f(x)} (g \wedge h)(x) \;=\; \bigwedge_{0<f(x)} g(x) \wedge \bigwedge_{0<f(x)} h(x) \;=\; \dot{f}(g) \wedge \dot{f}(h).$$

Thus $\dot{f}$ is a homogeneous fuzzy filter on X. $\Box$

Now we introduce the fuzzy neighborhood filter $\mathcal{N}(f)$ at a fuzzy set f.

Proposition 2.2 *For every fuzzy topological space (X, τ) and every fuzzy subset f of X, the supremum of the fuzzy neighborhood filters $\mathcal{N}(x)$*

$$\mathcal{N}(f) = \bigvee_{0<f(x)} \mathcal{N}(x) \tag{2.2}$$

is a fuzzy filter on X called a fuzzy neighborhood filter *at f.*

Proof. Since for all $\alpha \in L_0$ we have

$$\mathcal{N}(f)\,(\overline{\alpha}) \;=\; \bigwedge_{0<f(y)} \operatorname{int}_\tau \overline{\alpha}(y) \leq \bigwedge_{0<f(y)} \overline{\alpha}(y) \;=\; \alpha$$

and

$$\mathcal{N}(f)\,(\overline{1}) \;=\; \bigwedge_{0<f(y)} \operatorname{int}_\tau \overline{1}\,(y) \;=\; \bigwedge_{0<f(y)} \overline{1}(y) \;=\; 1.$$

Also we have

$$\begin{aligned}
\mathcal{N}(f)\,(g \wedge h) &= \bigwedge_{0<f(y)} \operatorname{int}_\tau (g \wedge h)\,(y) \;=\; \bigwedge_{0<f(y)} (\operatorname{int}_\tau g \wedge \operatorname{int}_\tau h)\,(y) \\
&= \bigwedge_{0<f(y)} \operatorname{int}_\tau g\,(y) \;\wedge \bigwedge_{0<f(y)} \operatorname{int}_\tau h\,(y) \;=\; \mathcal{N}(f)\,(g) \wedge \mathcal{N}(f)\,(h).
\end{aligned}$$

Hence, $\mathcal{N}(f)$ is a fuzzy filter on X.

Moreover, since $\operatorname{int}_\tau g\,(x) \leq g(x)$ for all $x \in X$ and $g \in L^X$, then $\bigwedge_{0<f(x)} \operatorname{int}_\tau g\,(x) \leq \bigwedge_{0<f(x)} g(x)$ and thus $\mathcal{N}(f)\,(g) \leq \dot{f}(g)$ for all $g \in L^X$. Hence, $\dot{f} \leq \mathcal{N}(f)$ and thus $\mathcal{N}(f)$ fulfills the condition (N1) of the fuzzy neighborhood filters.

Since for any $y \in X$ we have $\bigwedge_{0<f(y)} y \mapsto \bigwedge_{0<f(y)} \operatorname{int}_\tau g\,(y)$ represents the mapping $\operatorname{int}_\tau g$ and we have

$$\mathcal{N}(f)\,(\operatorname{int}_\tau g) \;=\; \bigwedge_{0<f(x)} (\operatorname{int}_\tau \operatorname{int}_\tau g)\,(x) \;=\; \bigwedge_{0<f(x)} \operatorname{int}_\tau g\,(x)$$

then

$$\mathcal{N}(f)\,(\bigwedge_{0<f(y)} y \mapsto \bigwedge_{0<f(y)} \operatorname{int}_\tau g\,(y)\,) \;=\; \mathcal{N}(f)\,(g).$$

Hence, $\mathcal{N}(f)$ fulfills (N2) of the fuzzy neighborhood filters. □

The following result is clear.

Remark 2.1 Note that the supremum of the empty set of fuzzy filters is the finest fuzzy filter and this means $\mathcal{N}(\overline{0}) \leq \dot{f}$ for all $f \in L^X$.

For any fuzzy topological space (X, τ) the fuzzy filters defined in (2.1) and (2.2) can be written as

$$\dot{f}\,(\lambda) = \bigwedge_{0<f(x)} \lambda(x), \tag{2.3}$$

$$\mathcal{N}(f)(\lambda) = \bigwedge_{0<f(x)} \mathcal{N}(x)(\lambda) = \bigwedge_{0<f(x)} \operatorname{int}_\tau \lambda(x), \tag{2.4}$$

for all $\lambda \in L^X$.

Lemma 2.1 *For all $f, g \in L^X$, we have*

$$f \leq g \text{ if and only if } \dot{f} \leq \dot{g}.$$

Proof. Since $f \leq g$ implies $f(x) \leq g(x)$ for all $x \in X$, then $\bigwedge_{0<f(x)} \lambda(x) \geq \bigwedge_{0<g(x)} \lambda(x)$ and this means $\dot{f}(\lambda) \geq \dot{g}(\lambda)$ for all $\lambda \in L^X$. Thus $\dot{f} \leq \dot{g}$.

Conversely, $\dot{f} \leq \dot{g}$ implies $\bigwedge_{0<f(x)} \lambda(x) \geq \bigwedge_{0<g(x)} \lambda(x)$ for all $\lambda \in L^X$. Suppose $g(x) \leq f(x)$ for some $x \in X$, then $0 < g(x)$ implies $0 < f(x)$ and hence $\bigwedge_{0<g(x)} \lambda(x) \geq \bigwedge_{0<f(x)} \lambda(x)$. Thus $\dot{g}(\lambda) \geq \dot{f}(\lambda)$ for all $\lambda \in L^X$, that is, $\dot{g} \leq \dot{f}$ which is a contradiction. Hence, $f(x) \leq g(x)$ for all $x \in X$ and then $f \leq g$. $\square$

Here a useful remark is given.

Remark 2.2 For all $x \in X$, $\alpha \in L_0$ and a fuzzy point x_α we have $\dot{x_\alpha} = \dot{x}$ and moreover, the fuzzy neighborhood filter $\mathcal{N}(x_\alpha)$ of x_α is itself the fuzzy neighborhood filter $\mathcal{N}(x)$ at x.

The fuzzy filters $\mathcal{N}(f)$ and $\dot{f}$ fulfill the following properties.

Lemma 2.2 *For all $f, g \in L^X$, the following properties are fulfilled:*

(1) $\dot{f} \leq \dot{g}$ *implies* $\mathcal{N}(g') \leq \dot{f}'$, *and consequently* $\mathcal{N}(f) \leq \dot{g}$ *implies* $\mathcal{N}(g') \leq \dot{f}'$.

(2) $f \leq g$ *implies* $\mathcal{N}(f) \leq \mathcal{N}(g)$.

(3) $\mathcal{N}(f \vee g) = \mathcal{N}(f) \wedge \mathcal{N}(g)$.

(4) $\mathcal{N}(f) \leq \dot{g}$ *implies* $f \leq g$.

(5) *If* $\mathcal{N}(f) \leq \dot{g}$, *then there is an* $h \in L^X$ *such that* $\mathcal{N}(f) \leq \dot{h}$ *and* $\mathcal{N}(h) \leq \dot{g}$.

Proof. $\dot{f} \leq \dot{g}$ implies $\dot{f} \leq \mathcal{N}(g)$ and hence for all $\lambda \in L^X$, we have

$$\bigwedge_{0<f(x)} \lambda(x) \geq \bigwedge_{0<g(y)} \operatorname{int}_\tau \lambda(y).$$

Thus

$$\bigwedge_{0<f'(x)} \lambda(x) \leq \bigwedge_{0<g'(y)} \operatorname{int}_\tau \lambda(y),$$

that is, $\dot{f}'(\lambda) \leq \mathcal{N}(g')(\lambda)$. Therefore $\mathcal{N}(g') \leq \dot{f}'$. Also, since $\mathcal{N}(f) \leq \dot{g}$ implies $\dot{f} \leq \dot{g}$, then $\mathcal{N}(f) \leq \dot{g}$ implies $\mathcal{N}(g') \leq \dot{f}'$ and hence (1) is fulfilled.

Since, $f \leq g$ implies $f(x) \leq g(x)$ for all $x \in X$, then

$$\bigwedge_{0<f(x)} \operatorname{int}_\tau \lambda(x) \geq \bigwedge_{0<g(x)} \operatorname{int}_\tau \lambda(x).$$

Hence, $\mathcal{N}(f)(\lambda) \geq \mathcal{N}(g)(\lambda)$ for all $\lambda \in L^X$ and this means $\mathcal{N}(f) \leq \mathcal{N}(g)$. Thus (2) is fulfilled.

From (2) we get $\mathcal{N}(f) \leq \mathcal{N}(f \vee g)$ and $\mathcal{N}(g) \leq \mathcal{N}(f \vee g)$, that is, $\mathcal{N}(f) \wedge \mathcal{N}(g) \leq \mathcal{N}(f \vee g)$. Now, for all $h \in L^X$ we have

$$\begin{aligned}
(\mathcal{N}(f) \wedge \mathcal{N}(g))(h) &= \bigvee_{k_1 \wedge k_2 \leq h} (\, \mathcal{N}(f)(k_1) \wedge \mathcal{N}(g)(k_2)\,) \\
&= \bigvee_{k_1 \wedge k_2 \leq h} (\, \bigwedge_{0<f(x)} \operatorname{int}_\tau k_1(x) \wedge \bigwedge_{0<g(y)} \operatorname{int}_\tau k_2(y)\,) \\
&\leq \bigvee_{k_1 \wedge k_2 \leq h} \bigwedge_{0<(f \vee g)(z)} \operatorname{int}_\tau (k_1 \wedge k_2)(z) \\
&\leq \bigwedge_{0<(f \vee g)(z)} \operatorname{int}_\tau h(z) \;=\; \mathcal{N}(f \vee g)(h).
\end{aligned}$$

Hence, $\mathcal{N}(f \vee g) \leq \mathcal{N}(f) \wedge \mathcal{N}(g)$ and therefore $\mathcal{N}(f \vee g) = \mathcal{N}(f) \wedge \mathcal{N}(g)$.

Since $\mathcal{N}(f) \leq \dot{g}$ implies $\dot{f} \leq \dot{g}$, then from Lemma 2.1 we get $f \leq g$ and hence (4) is satisfied.

Let $\mathcal{N}(f) \leq \dot{g}$. Then $\bigwedge_{0<f(x)} \operatorname{int}_\tau \lambda(x) \geq \bigwedge_{0<g(y)} \lambda(y)$ for all $\lambda \in L^X$, and hence there is an $h \in L^X$ such that

$$\bigwedge_{0<f(x)} \operatorname{int}_\tau \lambda(x) \geq \bigwedge_{0<h(z)} \lambda(z) \geq \bigwedge_{0<h(z)} \operatorname{int}_\tau \lambda(z) \geq \bigwedge_{0<g(y)} \lambda(y).$$

That means there is $h \in L^X$ such that $\mathcal{N}(f)\,(\lambda) \geq \dot{h}(\lambda)$ and $\mathcal{N}(h)\,(\lambda) \geq \dot{g}(\lambda)$ for all $\lambda \in L^X$. Thus $\mathcal{N}(f) \leq \dot{h}$ and $\mathcal{N}(h) \leq \dot{g}$. Hence, (5) is fulfilled. $\square$

A binary relation δ on L^X is called a *fuzzy proximity* ([12]) on X provided it fulfills the following conditions:

(P1) $f\overline{\delta}g$ implies $g\overline{\delta}f$, where $\overline{\delta}$ is the negation of δ.

(P2) $(f \vee g)\overline{\delta}h$ if and only if $f\overline{\delta}h$ and $g\overline{\delta}h$.

(P3) $f = \overline{0}$ or $g = \overline{0}$ implies $f\overline{\delta}g$ for all $f, g \in L^X$.

(P4) $f\overline{\delta}g$ implies $f \leq g'$.

(P5) If $f\overline{\delta}g$, then there is an $h \in L^X$ such that $f\overline{\delta}h$ and $h'\overline{\delta}g$.

Clearly, $(P1)$ and $(P2)$ imply the following condition:
(P2$'$) $h\overline{\delta}(f \vee g)$ if and only if $h\overline{\delta}f$ and $h\overline{\delta}g$.

A set X equipped with a fuzzy proximity δ on X is called a *fuzzy proximity space* (X, δ).

In the following proposition, the fuzzy proximity will be identified with the finer relation on fuzzy filters.

Proposition 2.3 *The binary relation δ on L^X which is defined by*

$$f\,\overline{\delta}\,g \quad \text{if and only if } \mathcal{N}(g) \leq \dot{f}'$$

is a fuzzy proximity on X.

Proof. From (1) in Lemma 2.2, it follows that $\mathcal{N}(g) \leq \dot{f}'$ implies $\mathcal{N}(f) \leq \dot{g}'$, and thus $f\,\overline{\delta}\,g$ implies $g\,\overline{\delta}\,f$. Hence, the condition (P1) of the fuzzy proximity is fulfilled.

Since $\mathcal{N}(f) \leq \mathcal{N}(f \vee g)$ and $\mathcal{N}(g) \leq \mathcal{N}(f \vee g)$ for all $f, g \in L^X$, then $\mathcal{N}(f \vee g) \leq \dot{h}'$ implies $\mathcal{N}(f) \leq \dot{h}'$ and $\mathcal{N}(g) \leq \dot{h}'$ for all $h \in L^X$. Thus, $h\,\overline{\delta}\,(f \vee g)$ implies $h\,\overline{\delta}\,f$ and $h\,\overline{\delta}\,g$. Conversely, if $\mathcal{N}(f) \leq \dot{h}'$ and $\mathcal{N}(g) \leq \dot{h}'$ for all $h \in L^X$, then from (3) in

Lemma 2.2 we have $\mathcal{N}(f) \wedge \mathcal{N}(g) = \mathcal{N}(f \vee g) \leq \dot{h}'$ and hence $h\,\overline{\delta}\,f$ and $h\,\overline{\delta}\,g$ imply $h\,\overline{\delta}\,(f \vee g)$. Thus (P2) is fulfilled.

From Remark 2.1, it follows $\mathcal{N}(\overline{0}) \leq \dot{f}'$ for all $f \in L^X$. This means $f\,\overline{\delta}\,\overline{0}$ for all $f \in L^X$. Hence, $f = \overline{0}$ or $g = \overline{0}$ implies $f\,\overline{\delta}\,g$. Therefore, (P3) is fulfilled.

From (4) in Lemma 2.2, it follows $\mathcal{N}(f) \leq \dot{g}'$ implies $f \leq g'$ and then $g\,\overline{\delta}\,f$ implies $f \leq g'$. That is, (P4) is fulfilled.

From (5) in Lemma 2.2 we have $\mathcal{N}(f) \leq \dot{g}'$ implies there is an $h \in L^X$ such that $\mathcal{N}(f) \leq \dot{h}$ and $\mathcal{N}(h) \leq \dot{g}'$. Hence, $g\,\overline{\delta}\,f$ implies there is an $h \in L^X$ such that $g\,\overline{\delta}\,h$ and $h'\,\overline{\delta}\,f$. Thus, (P5) holds. □

To each fuzzy proximity δ on a set X is associated a fuzzy topology τ_δ. The related interior and closure operators int_δ and cl_δ are given by

$$\mathrm{int}_\delta f \;=\; \bigvee_{f'\overline{\delta}g} g \tag{2.5}$$

and

$$\mathrm{cl}_\delta f \;=\; \bigwedge_{g'\overline{\delta}f} g \tag{2.6}$$

respectively, for all $f \in L^X$.

A fuzzy proximity δ on a set X is called *separated* if $x \neq y$ in X implies $x_\alpha \overline{\delta} y_\beta$ for all $\alpha, \beta \in L_0$ ([12]).

In the following we shall show that the associated fuzzy topology τ_δ of a fuzzy proximity δ is GT_0 if and only if δ is separated.

Proposition 2.4 *Let (X, δ) be a fuzzy proximity space and let τ_δ be the fuzzy topology associated to δ. Then*

$$\delta \;\; \textit{is} \text{ separated } \textit{if and only if } \tau_\delta \;\; \textit{is } GT_0.$$

Proof. Let δ be a separated fuzzy proximity and $x \neq y$ in X. Then $x_1\,\overline{\delta}\,y_1$ and this means, by Proposition 2.3 and Remark 2.2, that $\mathcal{N}(y) \leq \dot{x}'$. Thus $\mathrm{int}_\delta\, f(y) \geq$

$\bigwedge_{z\neq x} f(z)$ for all $f \in L^X$, taking $f = x_1'$ we get $\operatorname{int}_\delta x_1'(y) = 1$, $x_1'(x) = 0$ and hence there is $f = x_1' \in L^X$ such that $\operatorname{int}_\delta f(y) > f(x)$, that is, $\dot{x} \not\leq \mathcal{N}(y)$. Therefore, τ_δ is GT_0.

Now, let τ_δ be GT_0 and $x \neq y$ in X. Then $\dot{x} \not\leq \mathcal{N}(y)$ and this means there is $f \in L^X$ such that $\operatorname{int}_\delta f(y) > f(x)$. From (2.5) we get $\bigvee_{f' \overline{\delta} g} g(y) > f(x)$ and hence $f(x) < g(y)$ for all $g \in L^X$ with $f' \,\overline{\delta}\, g$, that is, $f(x) < g(y)$ for all $g \in L^X$ with $\mathcal{N}(g) \leq \dot{f}$. Taking $f = x_1'$ and $g = y_1$ we get $\mathcal{N}(y) \leq \dot{x}'$ and then by Proposition 2.3 we have $x_1 \,\overline{\delta}\, y_1$. Therefore $x_\alpha \,\overline{\delta}\, y_\beta$ for all $\alpha, \beta \in L_0$. □

In the following proposition we are going to show an important result which we need in the next part of this section.

Proposition 2.5 *If (X, δ) is a fuzzy proximity space, then*

$$f \,\overline{\delta}\, g \quad \text{if and only if} \quad \operatorname{cl}_\tau f \,\overline{\delta}\, \operatorname{cl}_\tau g$$

for all $f, g \in L^X$.

Proof. From Proposition 2.3, $\operatorname{cl}_\tau f \,\overline{\delta}\, \operatorname{cl}_\tau g$ implies $\mathcal{N}(\operatorname{cl}_\tau g) \leq (\dot{\operatorname{cl}_\tau f})'$. Since $f \leq \operatorname{cl}_\tau f$ and $\mathcal{N}(g) \leq \mathcal{N}(\operatorname{cl}_\tau g)$ for all $f, g \in L^X$, then $\mathcal{N}(g) \leq \dot{f}'$ and hence $f \,\overline{\delta}\, g$.

Conversely, since $f \,\overline{\delta}\, g$ implies $\mathcal{N}(g) \leq \dot{f}'$, then $\dot{g} \leq \mathcal{N}(f') \leq \mathcal{N}(\operatorname{cl}_\tau f')$. From (1) in Lemma 2.2 we have $\mathcal{N}(\operatorname{cl}_\tau f) \leq \dot{g}'$ and then $g \,\overline{\delta}\, \operatorname{cl}_\tau f$. Also, $g \,\overline{\delta}\, \operatorname{cl}_\tau f$ means that $\mathcal{N}(\operatorname{cl}_\tau f) \leq \dot{g}'$, that is, $(\dot{\operatorname{cl}_\tau f}) \leq \mathcal{N}(g')$ and once again by Lemma 2.2 we have

$$(\dot{\operatorname{cl}_\tau f}) \leq \mathcal{N}(g') \leq \mathcal{N}(\operatorname{cl}_\tau g')$$

implies $\mathcal{N}(\operatorname{cl}_\tau g) \leq (\dot{\operatorname{cl}_\tau f})'$. Thus $\operatorname{cl}_\tau f \,\overline{\delta}\, \operatorname{cl}_\tau g$. □

In the following we shall give another description of the fuzzy regular spaces.

Proposition 2.6 *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

(1) (X,τ) *is regular.*

(2) *For all* $x \in X$ *and* $f \in L^X$ *with* $\mathcal{N}(x) \leq \dot{f}$, *there exists* $g \in L^X$ *such that* $\mathcal{N}(x) \leq \dot{g}$ *and* $\mathcal{N}(\mathrm{cl}_\tau g) \leq \dot{f}$.

Proof. (1) $\Rightarrow$ (2): Let (X,τ) be a regular fuzzy topological space and let $\mathcal{N}(x) \leq \dot{f}$ for all $x \in X$ and $f \in L^X$. Then from (5) in Lemma 2.2 we have $g \in L^X$ such that

$$\mathcal{N}(x) \leq \dot{g} \text{ and } \mathcal{N}(g) \leq \dot{f}.$$

$\mathcal{N}(g) \leq \dot{f}$ means that

$$\bigwedge_{0<g(z)} \mathrm{int}_\tau \lambda\,(z) \;\geq\; \bigwedge_{0<f(y)} \lambda(y)$$

for all $\lambda \in L^X$. Since (X,τ) is regular, then $\mathrm{cl}_\tau\,\mathcal{N}(x) = \mathcal{N}(x)$ for all $x \in X$ and this means

$$\mathrm{int}_\tau \lambda\,(z) \;=\; \bigwedge_{0<g(z)} \;\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \mathrm{int}_\tau h\,(z) \;\geq\; \bigwedge_{0<f(y)} \lambda(y)$$

for all $\lambda \in L^X$, hence

$$\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \;\bigwedge_{0<g(z)} \mathrm{int}_\tau h\,(z) \;\geq\; \bigwedge_{0<f(y)} \lambda\,(y).$$

From that $g \leq \mathrm{cl}_\tau g$ for all $g \in L^X$ it follows

$$\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \;\bigwedge_{0<\mathrm{cl}_\tau g(z)} \mathrm{int}_\tau h\,(z) \;=\; \bigwedge_{0<\mathrm{cl}_\tau g(z)} \;\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \mathrm{int}_\tau h\,(z) \;\geq\; \bigwedge_{0<f(y)} \lambda\,(y).$$

Also, since

$$\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \mathrm{int}_\tau h\,(z) \;=\; \mathrm{cl}_\tau\,\mathcal{N}(z)\,(\lambda) \;=\; \mathcal{N}(z)\,(\lambda) \;=\; \mathrm{int}_\tau \lambda\,(z)$$

for all $z \in X$ and $\lambda \in L^X$, then

$$\bigwedge_{0<\mathrm{cl}_\tau g(z)} \mathrm{int}_\tau \lambda\,(z) \;\geq\; \bigwedge_{0<f(y)} \lambda(y).$$

Thus $\mathcal{N}(\mathrm{cl}_\tau g)\,(\lambda) \geq \dot{f}(\lambda)$ and hence $\mathcal{N}(\mathrm{cl}_\tau g) \leq \dot{f}$. That is, $\mathcal{N}(x) \leq \dot{f}$ implies there is $g \in L^X$ such that $\mathcal{N}(x) \leq \dot{g}$ and $\mathcal{N}(\mathrm{cl}_\tau g) \leq \dot{f}$, and therefore (2) holds.

(2) $\Rightarrow$ (1): Let (2) be fulfilled and let $x \notin F = \mathrm{cl}_\tau F$ in X. Then $x \in F'$ and from Lemma 2.1 and from (2) in Lemma 2.2 we get $\mathcal{N}(x) \leq \dot{\chi_{F'}}$ and then there is $g \in L^X$ such that $\mathcal{N}(x) \leq \dot{g}$ and $\mathcal{N}(g) \leq \mathcal{N}(\mathrm{cl}_\tau g) \leq \dot{\chi_{F'}}$. From (1) in Lemma 2.2 we have $\mathcal{N}(F) \leq \dot{g'}$ and hence

$$\mathcal{N}(x)\,(\lambda) \wedge \mathcal{N}(F)\,(\mu) \;\geq\; \dot{g}\,(\lambda) \wedge \dot{g'}\,(\mu) \;=\; \bigwedge_{0<g(m)} \lambda(m) \;\wedge\; \bigwedge_{0<g'(n)} \mu(n)$$

for all $\lambda, \mu \in L^X$. Taking $g = x_1 \vee y_1$ for $x \neq y \in F'$ we get

$$\mathcal{N}(x)\,(\lambda) \wedge \mathcal{N}(F)\,(\mu) \geq \bigwedge_{0<(x_1\vee y_1)(m)} \lambda(m) \;\wedge\; \bigwedge_{0<(x_1\vee y_1)'(n)} \mu(n)$$

for all $\lambda, \mu \in L^X$. Since for $\lambda = (x_1 \vee y_1)$ and $\mu = (x_1 \vee y_1)'$ we have $\sup(\lambda \wedge \mu) = 0$ and $\mathcal{N}(x)\,(\lambda) \wedge \mathcal{N}(F)\,(\mu) > 0$, then $\mathcal{N}(x) \wedge \mathcal{N}(F)$ does not exist and hence (X, τ) is regular. □

In the next proposition will be shown that for each fuzzy proximity is associated a regular, in our sense, fuzzy topology.

Proposition 2.7 *Let δ be a fuzzy proximity on a set X. Then the associated fuzzy topology τ_δ is regular.*

Proof. Let $x \in X$ and $f \in L^X$ with $\mathcal{N}(x) \leq \dot{f}$. Then $f' \,\overline{\delta}\, x_1$, and from (P5) we have there is $g \in L^X$ such that $f' \,\overline{\delta}\, g$ and $g' \,\overline{\delta}\, x_1$. By means of Proposition 2.5 we have $\mathrm{cl}_\delta f' \,\overline{\delta}\, \mathrm{cl}_\delta g$ and hence $\mathcal{N}(\mathrm{cl}_\delta g) \leq (\mathrm{cl}_\delta \dot{f'})' \leq \dot{f}$, and $\mathcal{N}(x) \leq \dot{g}$. Therefore, $\mathcal{N}(x) \leq \dot{f}$ implies there exists $g \in L^X$ such that $\mathcal{N}(x) \leq \dot{g}$ and $\mathcal{N}(\mathrm{cl}_\delta g) \leq \dot{f}$ and thus (X, τ_δ) is regular. □

In the following we shall give an equivalent form for the fuzzy normal spaces.

Proposition 2.8 *Let (X, τ) be a fuzzy topological space. Then the following statements are equivalent:*

(1) *(X, τ) is normal.*

(2) *For all* $A \subseteq X$ *with* $\mathrm{cl}_\tau A = A$ *and all* $f \in L^X$ *with* $\mathcal{N}(A) \leq \dot{f}$*, there exists* $g \in L^X$ *such that* $\mathcal{N}(A) \leq \dot{g}$ *and* $\mathcal{N}(\mathrm{cl}_\tau g) \leq \dot{f}$*.*

Proof. (1) ⇒ (2): Let (X, τ) be a normal fuzzy topological space and let $\mathcal{N}(A) \leq \dot{f}$ for all $A \subseteq X$ with $A = \mathrm{cl}_\tau A$ in X and all $f \in L^X$. Then from (5) in Lemma 2.2 we have $g \in L^X$ such that

$$\mathcal{N}(A) \leq \dot{g} \text{ and } \mathcal{N}(g) \leq \dot{f}.$$

$\mathcal{N}(g) \leq \dot{f}$ means that

$$\bigwedge_{0<g(z)} \mathrm{int}_\tau \lambda\,(z) \;\geq \bigwedge_{0<f(y)} \lambda\,(y)$$

for all $\lambda \in L^X$. Since (X, τ) is normal, then $\mathrm{cl}_\tau \mathcal{N}(A) = \mathcal{N}(A)$, that is,

$$\bigwedge_{x\in A} \mathrm{int}_\tau \lambda\,(x) \;=\; \bigvee_{\mathrm{cl}_\tau h \leq \lambda} \;\bigwedge_{x\in A} \mathrm{int}_\tau h\,(x)$$

for all $\lambda \in L^X$. But $\mathcal{N}(A) \leq \dot{g}$ implies $\dot{A} \leq \dot{g}$ and from Lemma 2.1 we get $A \subseteq S_0 g$ and then

$$\bigwedge_{x\in A} \mathrm{int}_\tau \lambda\,(x) \;\geq \bigwedge_{0<g(x)} \mathrm{int}_\tau \lambda\,(x)$$

for all $\lambda \in L^X$. Hence, similarly as in the first direction in Proposition 2.6 we have $\mathcal{N}(g) \leq \dot{f}$ implies

$$\begin{aligned}\bigwedge_{x\in A} \mathrm{int}_\tau \lambda\,(x) \;&=\; \bigwedge_{0<g(x)} \;\bigwedge_{x\in A} \mathrm{int}_\tau \lambda\,(x) \;=\; \bigwedge_{0<g(x)} \;\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \;\bigwedge_{x\in A} \mathrm{int}_\tau h\,(x)\\ &\geq\; \bigwedge_{0<g(x)} \mathrm{int}_\tau \lambda\,(z) \;\geq\; \bigwedge_{0<f(y)} \lambda\,(y),\end{aligned}$$

which means, from that $g \leq \mathrm{cl}_\tau g$,

$$\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \;\bigwedge_{0<\mathrm{cl}_\tau g(x)} \;\bigwedge_{x\in A} \mathrm{int}_\tau h\,(x) \;\geq \bigwedge_{0<f(y)} \lambda(y)$$

and this means

$$\bigwedge_{0<\mathrm{cl}_\tau g(x)} \;\bigvee_{\mathrm{cl}_\tau h \leq \lambda} \;\bigwedge_{x\in A} \mathrm{int}_\tau h\,(x) \;=\; \bigwedge_{0<\mathrm{cl}_\tau g(x)} \;\bigwedge_{x\in A} \mathrm{int}_\tau \lambda\,(x) \;\geq \bigwedge_{0<f(y)} \lambda(y).$$

Hence, $\mathcal{N}(\mathrm{cl}_\tau g)\,(\lambda) \geq \dot{f}(\lambda)$ for all $\lambda \in L^X$ and thus $\mathcal{N}(\mathrm{cl}_\tau g) \leq \dot{f}$. Therefore $\mathcal{N}(A) \leq \dot{f}$ implies there is $g \in L^X$ such that $\mathcal{N}(A) \leq \dot{g}$ and $\mathcal{N}(\mathrm{cl}_\tau h) \leq \dot{f}$ and thus (2) holds.

(2) $\Rightarrow$ (1): Let (2) be hold and let A, B be two disjoint closed subsets of X. Then $A \subseteq B'$ and hence from Lemma 2.1 and (2) in Lemma 2.2 we get $\mathcal{N}(A) \leq \chi_{B'}$. From (2) in this proposition we have $g \in L^X$ such that $\mathcal{N}(A) \leq \dot{g}$ and $\mathcal{N}(\mathrm{cl}_\tau\, g) \leq \chi_{B'}$, that is, $\mathcal{N}(A) \leq \dot{g}$ and $\mathcal{N}(B) \leq (\mathrm{cl}_\tau^{\cdot}\, g)' \leq \dot{g}'$ which means

$$\mathcal{N}(A)\,(\lambda) \wedge \mathcal{N}(B)\,(\mu) \geq \; \dot{g}(\lambda) \wedge \dot{g}'(\mu)$$

for all $\lambda, \mu \in L^X$. Taking $g = (\chi_A \vee x_1)$ for $x \in B' \setminus A$. Then for $\lambda = g$ and $\mu = g'$ we have $\sup(\lambda \wedge \mu) = 0$ and $\mathcal{N}(A)\,(\lambda) \wedge \mathcal{N}(B)\,(\mu) > \sup(\lambda \wedge \mu)$ and therefore (X, τ) is normal. □

Now, we are going to show another important relation between the fuzzy proximity and the fuzzy normal spaces in our sense.

Proposition 2.9 *If (X, τ) is a normal fuzzy topological space, then the binary relation δ on X defined by*

$$f\,\overline{\delta}\,g \iff \mathcal{N}(\mathrm{cl}_\tau\, f) \leq (\mathrm{cl}_\tau^{\cdot}\, g)' \tag{2.7}$$

is a fuzzy proximity on X. Conversely, in a fuzzy proximity space (X, δ) with δ fulfills (2.7) the fuzzy topological space (X, τ_δ) is normal.

Proof. Let (X, τ) be a fuzzy topological space and let δ be a binary relation defined by (2.7). $f\,\overline{\delta}\,g$ implies $\mathcal{N}(\mathrm{cl}_\tau\, f) \leq (\mathrm{cl}_\tau^{\cdot}\, g)'$ and from (1) in Lemma 2.2 we get $\mathcal{N}(\mathrm{cl}_\tau\, g) \leq (\mathrm{cl}_\tau^{\cdot}\, f)'$ and then $g\,\overline{\delta}\,f$. Hence, the condition (P1) of the fuzzy proximity is fulfilled.

Let $(f \vee g)\,\overline{\delta}\,h$. Then

$$\mathcal{N}(\mathrm{cl}_\tau\, f \vee \mathrm{cl}_\tau\, g) \;=\; \mathcal{N}(\mathrm{cl}_\tau\,(f \vee g)) \leq (\mathrm{cl}_\tau^{\cdot}\, h)'.$$

From Lemma 2.2 it follows $\mathcal{N}(\mathrm{cl}_\tau\, f) \leq (\mathrm{cl}_\tau^{\cdot}\, h)'$ and $\mathcal{N}(\mathrm{cl}_\tau\, g) \leq (\mathrm{cl}_\tau^{\cdot}\, h)'$. Thus $(f \vee g)\,\overline{\delta}\,h$ implies $f\,\overline{\delta}\,h$ and $g\,\overline{\delta}\,h$. Also, from Lemma 2.2 we have $\mathcal{N}(\mathrm{cl}_\tau\, f) \leq \mathrm{cl}_\tau^{\cdot} h'$ and

$\mathcal{N}(\mathrm{cl}_\tau\, g) \leq \dot{\mathrm{cl}_\tau h'}$ implies

$$\mathcal{N}(\mathrm{cl}_\tau\, f \vee \mathrm{cl}_\tau\, g) \;=\; \mathcal{N}(\mathrm{cl}_\tau\, f) \wedge \mathcal{N}(\mathrm{cl}_\tau\, g) \leq \dot{\mathrm{cl}_\tau h'},$$

that is, $f\overline{\delta}h$ and $g\overline{\delta}h$ implies $(f \vee g)\,\overline{\delta}\,h$. Therefore, (P2) is fulfilled.

Since $\mathcal{N}(\overline{0})$ is the finest fuzzy filter on X and from that $\mathrm{cl}_\tau\,\overline{0} = \overline{0}$ we get $\mathcal{N}(\overline{0}) = \mathcal{N}(\mathrm{cl}_\tau\,\overline{0}) \leq (\dot{\mathrm{cl}_\tau f})'$ for all $f \in L^X$. Thus $\overline{0}\,\overline{\delta}\,f$ for all $f \in L^X$, in other wards $f = \overline{0}$ or $g = \overline{0}$ implies $f\,\overline{\delta}\,g$ and then (P3) holds.

Since $f\,\overline{\delta}\,g$ implies $\mathcal{N}(\mathrm{cl}_\tau\, f) \leq (\dot{\mathrm{cl}_\tau g})'$ which means $(\dot{\mathrm{cl}_\tau f}) \leq (\dot{\mathrm{cl}_\tau g})'$. From Lemma 2.1 we get $f \leq \mathrm{cl}_\tau\, f \leq (\mathrm{cl}_\tau\, g)' \leq g'$ and hence (P4) is satisfied.

Let $f\,\overline{\delta}\,g$. Then $\mathcal{N}(\mathrm{cl}_\tau\, f) \leq (\dot{\mathrm{cl}_\tau g})'$. Taking $F = S_0(\mathrm{cl}_\tau\, f)$, then we get $F = \mathrm{cl}_\tau\, F$ in X and then $f\,\overline{\delta}\,g$ implies $\mathcal{N}(F) \leq (\dot{\mathrm{cl}_\tau g})'$. From the normality condition listed in Proposition 2.8, there is a fuzzy set $h' \in L^X$ with arbitrary choice such that $\mathcal{N}(F) = \mathcal{N}(\mathrm{cl}_\tau\, f) \leq (\dot{\mathrm{cl}_\tau h})' \leq (\dot{h'})$ such that $\mathcal{N}(\mathrm{cl}_\tau\, h') \leq (\dot{\mathrm{cl}_\tau g})'$. Hence, we get $h \in L^X$ such that $\mathcal{N}(\mathrm{cl}_\tau\, f) \leq (\dot{\mathrm{cl}_\tau h})'$ and $\mathcal{N}(\mathrm{cl}_\tau\, h') \leq (\dot{\mathrm{cl}_\tau g})'$, which means that $f\,\overline{\delta}\,h$ and $h'\,\overline{\delta}\,g$. Thus (P5) is fulfilled.

Conversely, let A and B are two disjoint closed subsets of X. Then $A \subseteq B'$ and then $\dot{\chi_A} \leq \dot{\chi_{B'}} = \dot{\chi'_B}$, and hence from (1) in Lemma 2.2 we have $\mathcal{N}(\chi_B) \leq \dot{\chi'_A}$. Since A, B are closed, then

$$\mathcal{N}(\mathrm{cl}_\delta\, \chi_B) = \mathcal{N}(\chi_B) \leq \dot{\chi'_A} = (\dot{\mathrm{cl}_\delta\, \chi_A})'$$

which means that $\chi_A\,\overline{\delta}\,\chi_B$. From (P5) there exists $g \in L^X$ such that

$$\mathcal{N}(\chi_B) \;=\; \mathcal{N}(B) \leq \dot{g} \text{ and } \mathcal{N}(g) \leq \dot{\chi'_A} \;=\; \dot{A}'$$

and again from (1) in Lemma 2.2 we have $\mathcal{N}(A) \leq \dot{g'}$. Hence,

$$\mathcal{N}(B)\,(\lambda) \wedge \mathcal{N}(A)\,(\mu) \geq \dot{g}(\lambda) \wedge \dot{g'}(\mu)$$

for all $\lambda, \mu \in L^X$. Taking $g = \chi_B \vee x_1 \in L^X$ for $x \in A' \setminus B$. Then if we take $\lambda = g = \chi_B \vee x_1$ and $\mu = g' = (\chi_B \vee x_1)'$, we get $\sup\,(\lambda \wedge \mu) = 0$ and

$$\mathcal{N}(B)\,(\lambda) \wedge \mathcal{N}(A)\,(\mu) \geq 0.$$

Thus, we get $\lambda, \mu \in L^X$ such that

$$\mathcal{N}(B)(\lambda) \wedge \mathcal{N}(A)(\mu) \geq \sup(\lambda \wedge \mu)$$

and therefore (X, τ_δ) is normal. □

3. G-Compact Spaces

The notion of G-Compactness is defined by means of the fuzzy filters and therefore it will be suitable to study here the relation between this notion and our fuzzy separation axioms GT_i.

Let $\mathcal{M}$ be a fuzzy filter on a set X. The element $x \in X$ is called a *cluster point* of $\mathcal{M}$ if the infimum $\mathcal{M} \wedge \mathcal{N}(x)$ of $\mathcal{M}$ and the fuzzy neighborhood filter $\mathcal{N}(x)$ at x exists, equivalently if there exists a fuzzy filter $\mathcal{K}$ finer than $\mathcal{M}$ which converges to x, that is, $\mathcal{K} \leq \mathcal{N}(x)$. Notice that any convergent fuzzy filter has a cluster point.

A fuzzy topological space (X, τ) is called *G-compact* if every fuzzy filter on X has a cluster point in X ([9]).

Let (X, τ) be a fuzzy topological space. Then a subset A of X is called *closed* with respect to τ if $\mathcal{M} \leq \mathcal{N}(x)$ implies $x \in A$ for some $\mathcal{M} \in \mathcal{F}_L A$.

Our GT_2-spaces fulfill the following result.

Proposition 3.1 *Every G-compact subset of GT_2-space is closed.*

Proof. Let (X, τ) be a GT_2-space and let A be a G-compact subset of X. Then for all $\mathcal{M} \in \mathcal{F}_L A$, there exists $\mathcal{K} \leq \mathcal{M}$, $\mathcal{K} \in \mathcal{F}_L A$ such that $\mathcal{K} \leq \mathcal{N}(x)$ for some $x \in A$. Since $\mathcal{K} \in \mathcal{F}_L A \subseteq \mathcal{F}_L X$ and (X, τ) is GT_2, then $\mathcal{K} \leq \mathcal{N}(x)$ and $\mathcal{K} \leq \mathcal{N}(y)$ imply $y = x$, that is, for some $\mathcal{K} \in \mathcal{F}_L A$ with $\mathcal{K} \leq \mathcal{N}(y)$ we get $y \in A$. Hence, A is closed. □

In the following proposition we show another property of GT_2-spaces.

Proposition 3.2 *Let (X, τ) be a GT_2-space. Then any two disjoint G-compact subsets A and B of X have fuzzy neighborhood filters $\mathcal{N}(A)$ and $\mathcal{N}(B)$ such that $\mathcal{N}(A) \wedge \mathcal{N}(B)$ does not exist.*

Proof. Let A and B be two disjoint G-compact subsets of X. Then for all $\mathcal{K} \in \mathcal{F}_L A$, there exists $\mathcal{M} \leq \mathcal{K}$, $\mathcal{M} \in \mathcal{F}_L A$ such that $\mathcal{M} \leq \mathcal{N}(x)$ for some $x \in A$ and for all $\mathcal{L} \in \mathcal{F}_L B$, there exists $\mathcal{N} \leq \mathcal{L}$, $\mathcal{N} \in \mathcal{F}_L B$ such that $\mathcal{N} \leq \mathcal{N}(y)$ for some $y \in B$. Since $\mathcal{F}_L A \subseteq \mathcal{F}_L X$ and $\mathcal{F}_L B \subseteq \mathcal{F}_L X$, then we can say that

$$\mathcal{M} \leq \mathcal{N}(x) \leq \mathcal{N}(A), \ \mathcal{N} \leq \mathcal{N}(y) \leq \mathcal{N}(B)$$

and there is $\mathcal{W} = (\mathcal{M} \wedge \mathcal{N}) \in \mathcal{F}_L X$ such that $\mathcal{W} \leq \mathcal{N}(x)$ and $\mathcal{W} \leq \mathcal{N}(y)$ for some $x \in A$ and some $y \in B$. But (X, τ) is GT_2-space and hence $x = y$ which contradicts that A and B are disjoint. Hence, for every $\mathcal{V} \in \mathcal{F}_L X$ we get $\mathcal{V} \not\leq \mathcal{N}(A)$ or $\mathcal{V} \not\leq \mathcal{N}(B)$ which means that $\mathcal{N}(A) \wedge \mathcal{N}(B)$ does not exist. Thus A and B can be separated by two disjoint neighborhoods. $\Box$

The notion of G-compactness fulfills the following property which will be used to prove the important result given in Proposition 3.4.

Proposition 3.3 *Every closed subset of G-compact space (X, τ) is G-compact.*

Proof. Let A be a closed subset of X and (X, τ) be G-compact space, and let $\mathcal{M} \in \mathcal{F}_L A$. Then $\mathcal{M} \leq \mathcal{N}(x)$ implies $x \in A$. Since, $\mathcal{F}_L A \subseteq \mathcal{F}_L X$, then $\mathcal{M} \in \mathcal{F}_L X$ and hence there exists $\mathcal{K} \leq \mathcal{M}$ such that $\mathcal{K} \leq \mathcal{N}(x)$ and

$$\mathcal{K} \leq \mathcal{M} \in \mathcal{F}_L A \subseteq \mathcal{F}_L X$$

which means that $\mathcal{K} \in \mathcal{F}_L A$. Thus for $\mathcal{M} \in \mathcal{F}_L A$ we get $\mathcal{K} \leq \mathcal{M}$ such that $\mathcal{K} \leq \mathcal{N}(x)$, $x \in A$. Hence A is G-compact. $\Box$

The following proposition introduces an important property of G-compact GT_2-spaces.

Proposition 3.4 *Every G-compact GT_2-space is GT_4-spaces.*

Proof. The proof follows directly from Propositions 3.2 and 3.3. $\square$

Lemma 3.1 *If τ_1 and τ_2 are fuzzy topologies on a set X, τ_1 is finer than τ_2 and (X, τ_1) is G-compact, then (X, τ_2) is also G-compact.*

Proof. Let $\mathcal{N}_{\tau_1}(x)$ and $\mathcal{N}_{\tau_2}(x)$ be the fuzzy neighborhood filters at x with respect to τ_1 and τ_2, respectively. Since τ_1 is finer than τ_2 implies $\mathcal{N}_{\tau_1}(x) \leq \mathcal{N}_{\tau_2}(x)$ for all $x \in X$, then for any $\mathcal{M} \in \mathcal{F}_L X$ with $\mathcal{M} \leq \mathcal{N}_{\tau_1}(x)$ we get $\mathcal{M} \leq \mathcal{N}_{\tau_2}(x)$. Hence, (X, τ_1) is G-compact implies (X, τ_2) is G-compact. $\square$

Proposition 3.5 [1] *For any GT_2-space (X, τ) and any fuzzy topology σ on X which is finer than τ, we have (X, σ) is also GT_2-space.*

Proposition 3.6 *Let τ_1 and τ_2 be fuzzy topologies on a set X with τ_1 be finer than τ_2, (X, τ_1) be G-compact space and let (X, τ_2) be GT_2-space. Then τ_1 is equivalent to τ_2.*

Proof. From Proposition 3.5 we get (X, τ_1) is also GT_2-space, and from Lemma 3.1 we have (X, τ_2) is also G-compact space. Then we can find the identity mapping $\mathrm{id}_X : (X, \tau_1) \to (X, \tau_2)$ which is a bijective fuzzy continuous mapping and open, that is, a homeomorphism. Hence, (X, τ_1) is equivalent to (X, τ_2). $\square$

4. Fuzzy Uniform Spaces

In this section we study the relation between the fuzzy uniform spaces introduced in [10] and the GT_i-spaces.

By a *fuzzy relation* on a set X we mean a mapping $u : X \times X \to L$, that is, a fuzzy subset of $X \times X$. For each fuzzy relation u on X, the *inverse* u^{-1} of u is the fuzzy relation on X defined by $u^{-1}(x,y) = u(y,x)$ for all $x, y \in X$. Let $\mathcal{U}$ be fuzzy filter on $X \times X$. The *inverse* $\mathcal{U}^{-1}$ of $\mathcal{U}$ is a fuzzy filter on $X \times X$ defined by

$$\mathcal{U}^{-1}(u) \;=\; \mathcal{U}(u^{-1})$$

for all $u \in L^{X\times X}$.

The *composition* $v \circ u$ of two fuzzy relations u and v on X is the fuzzy relation on X defined by

$$(v \circ u)(x,y) \;=\; \bigvee_{z\in X} (\, u(x,z) \wedge v(z,y))$$

for all $x, y \in X$.

For each pair (x,y) of elements x, y of X, the mapping $(x,y)^{\bullet} : L^{X\times X} \to L$ defined by

$$(x,y)^{\bullet}(u) \;=\; u(x,y)$$

for all $u \in L^{X\times X}$ is a homogeneous fuzzy filter on $X \times X$.

Let $\mathcal{U}$ and $\mathcal{V}$ be fuzzy filters on $X \times X$ such that $(x,y)^{\bullet} \le \mathcal{U}$ and $(y,z)^{\bullet} \le \mathcal{V}$ hold for some $x, y, z \in X$. Then the *composition* $\mathcal{V} \circ \mathcal{U}$ of $\mathcal{U}$ and $\mathcal{V}$ is a fuzzy filter on $X \times X$ defined by

$$(\mathcal{V} \circ \mathcal{U})(w) \;=\; \bigvee_{v\circ u\le w} (\mathcal{U}(u) \wedge \mathcal{V}(v))$$

for all $w \in L^{X\times X}$ ([10]).

By a *fuzzy uniform structure* $\mathcal{U}$ on a set X ([10]) we mean a fuzzy filter on $X \times X$ such that the following conditions are fulfilled:

(U1) $(x,x)^{\bullet} \le \mathcal{U}$ for all $x \in X$.

(U2) $\mathcal{U} = \mathcal{U}^{-1}$.

(U3) $\mathcal{U} \circ \mathcal{U} \le \mathcal{U}$.

The pair $(X,\mathcal{U})$ is called *fuzzy uniform space.*

To each fuzzy uniform structure $\mathcal{U}$ on X is associated a stratified fuzzy topology $\tau_{\mathcal{U}}$. The related interior operator $\text{int}_{\mathcal{U}}$ is given by

$$\text{int}_{\mathcal{U}} f(x) \;=\; \mathcal{U}[\dot{x}](f) \tag{4.1}$$

for all $x \in X$ ([4, 10]).

A fuzzy uniform structure $\mathcal{U}$ on X is called *separated* if for all $x, y \in X$ with $x \neq y$ there is $u \in L^{X\times X}$ such that $\mathcal{U}(u) = 1$ and $u(x, y) = 0$. The space $(X,\mathcal{U})$ is called *separated fuzzy uniform space.*

Proposition 4.1 *Let X be a set, $\mathcal{U}$ a fuzzy uniform structure on X and $\tau_{\mathcal{U}}$ the fuzzy topology induced by $\mathcal{U}$. Then*

$(X,\mathcal{U})$ separated if and only if $(X, \tau_{\mathcal{U}})$ is GT_0-space.

Proof. For $x \neq y$ if $(X,\mathcal{U})$ is separated, then there is $u \in L^{X\times X}$ such that $\mathcal{U}(u) = 1$ and $u(x, y) = 0$. Let $f = u[y_1]$ for which

$$f(x) = u[y_1](x) = \bigvee_{z\in X} u(z, x) \wedge y_1(z) = 0$$

and

$$\text{int}_{\mathcal{U}} f(y) = \mathcal{U}[\dot{y}](f) = \bigvee_{u[g]\leq f} \mathcal{U}(u) \wedge g(y) = 1,$$

that is, there is $f = u[y_1] \in L^X$ such that $f(x) < \text{int}_{\mathcal{U}} f(y)$ and hence $\dot{x} \nleq \mathcal{N}_{\mathcal{U}}(y)$, where $\mathcal{N}_{\mathcal{U}}(y)$ is the fuzzy neighborhood filter of the space $(X, \tau_{\mathcal{U}})$ at y. This means $(X, \tau_{\mathcal{U}})$ is GT_0-space.

Now let $(X, \tau_{\mathcal{U}})$ be GT_0-space and $x \neq y$ in X. Then there is $f \in L^X$ such that $f(x) < \text{int}_{\mathcal{U}} f(y)$ and this means $\bigvee_{u[g]\leq f} \mathcal{U}(u) \wedge g(y) > f(x)$. Hence, there is

$$u(x, y) = \begin{cases} \text{int}_{\mathcal{U}} f(y) & \text{if } x = y, \\ f(x) & \text{if } x \neq y \end{cases}$$

for which $u(x, y) = 0$ and $\mathcal{U}(u) = 1$. Thus, $(X,\mathcal{U})$ is separated. □

References

[1] F. Bayoumi and I. Ibedou, *GT_i-spaces*, I, The Journal of Egyptian society of Mathematics, Accepted in 12 December 2001.

[2] F. Bayoumi and I. Ibedou, *GT_i-spaces*, II, The Journal of Egyptian Society of Mathematics, Accepted in 12 December 2001.

[3] F. Bayoumi and I. Ibedou, *On GT_i-spaces*, Jour. of Inst. of Math. and Comp. Sci., Vol. 14, No. 3 (2001) 187 – 199.

[4] F. Bayoumi, *On initial and final fuzzy uniform structures*, Fuzzy Sets and Systems, **133** (2003) 299 – 319.

[5] C. L. Chang, *Fuzzy topological spaces*, J. Math. Anal. Appl., **24** (1968), 182 - 190.

[6] Á. Császár, *General Topology*, Akadémiai Kiadó, Budapest 1978.

[7] P. Eklund and W. Gähler, *Fuzzy filter functors and convergence*, in: Applications of Category Theory To Fuzzy Subsets, Kluwer Academic Publishers, Dordrecht et al. (1992), 109 - 136.

[8] W. Gähler, *The general fuzzy filter approach to fuzzy topology*, I, Fuzzy Sets and Systems, **76** (1995), 205 - 224.

[9] W. Gähler, *The general fuzzy filter approach to fuzzy topology*, II, Fuzzy Sets and Systems, **76** (1995), 225 - 246.

[10] W. Gähler, F. Bayoumi, A. Kandil and A. Nouh, *The theory of global fuzzy neighborhood structures.* (III), *Fuzzy uniform structures*, Fuzzy Sets and Systems, **98** (1998), 175 - 199.

[11] J. A. Goguen, *L-fuzzy sets*, J. Math. Anal. Appl., **18** (1967), 145 - 174.

[12] A. K. Katsaras, *Fuzzy proximities and fuzzy completely regular spaces*, I, An. Ştiinto Univ. Iasşi Sect. Ia Math. (N.S.) **26** (1980), 31 - 41.

[13] R. Lowen, *Convergence in fuzzy topological spaces*, General Topology and Appl., **10** (1979), 147 - 160.

[14] A. Weil, *Sur les espaces à structures uniformes et sur la topologie générale*, Hermann, Paris, 1937.

[15] L. A. Zadeh, *Fuzzy sets*, Information and Control, **8** (1965), 338 - 353.

[16] M. S. Elnaschie, *Quantum gravity from descriptive set theory*, C. S & F (2003) in press.